Contest # 4 — *Answers & Solutions* — 1/10/17

Problem 4-1

Each of the 2017 terms of $2 + 4 + 6 + \ldots$ is 1 more than the corresponding term of $1 + 3 + 5 + \ldots$, so the sum of the first 2017 even integers exceeds that of the first 2017 odd integers by $\boxed{2017}$.

Problem 4-2

By trial and error, 3 does not work since $1+3 = 4$. Continuing, the sum of the divisors of 4 is $1 + 2 + 4 = 7$, so the least positive integer greater than 2 whose divisor-sum is a prime is $\boxed{4}$.

[NOTE: Positive integers with a prime divisor-sum must be expressible as a power of a prime. Examples include $3^2 = 9$, $2^4 = 16$, and $5^2 = 25$, but **not** 7^2.]

Problem 4-3

The probability of getting

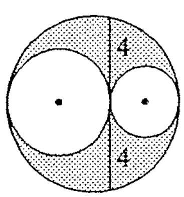
A) no bullseye in 1 try $= \frac{1}{2}$

B) at least 1 bullseye in 2 tries $= \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$

C) at least 2 bullseyes in 3 tries $= 3 \times \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$.

The most likely of these options is $\boxed{B}$.

Problem 4-4



The larger (and other) unshaded circles have respective radius-lengths 3 and r. The shaded region's area = the area of circle C − the sum of the areas of the unshaded circles $= \pi(3+r)^2 - (9\pi + \pi r^2) = 6\pi r$. When 2 chords intersect inside a circle, the product of the segment-lengths of one = the product of the segment-lengths of the other. Consequently, $6 \times 2r = 4 \times 4$, $r = 4/3$, and $6\pi r = \boxed{8\pi}$.

[NOTE: It's not too difficult to prove the surprising fact that the area of the shaded region is independent of the radius-lengths of the unshaded circles, as long as the product of their radius-lengths is 4.]

Problem 4-5

Method I: The equation is a quadratic in $(\log_{10}x)$ whose discriminant $= 36 + 4a \geq 0$ for $(\log_{10}x)$ to be a real number. The least such real a is $\boxed{-9}$.

Method II: Complete the square to get $(\log_{10}x)^2 + 6\log_{10}x + 9 = a + 9$, so $(\log_{10}x + 3)^2 = a + 9$. Since $\log_{10}x$ can be any real number, the left side is the square of a real number, so the left side is ≥ 0. Therefore, the left side = the right side $= a + 9 \geq 0$. The least such real a is -9.

Problem 4-6

There are 1999 terms, and the term in the middle is

$$\frac{4^{\frac{1000}{2000}}}{4^{\frac{1000}{2000}} + 2} = \frac{2}{2+2} = 0.5.$$

By pairing the n^{th} and the $(2000-n)^{\text{th}}$ terms, the sum of the other 1998 terms is

$$\left(\frac{4^{\frac{1}{2000}}}{4^{\frac{1}{2000}}+2} + \frac{4^{\frac{1999}{2000}}}{4^{\frac{1999}{2000}}+2}\right) + \left(\frac{4^{\frac{2}{2000}}}{4^{\frac{2}{2000}}+2} + \frac{4^{\frac{1998}{2000}}}{4^{\frac{1998}{2000}}+2}\right) + \ldots + \left(\frac{4^{\frac{999}{2000}}}{4^{\frac{999}{2000}}+2} + \frac{4^{\frac{1001}{2000}}}{4^{\frac{1001}{2000}}+2}\right).$$

As long as no denominator is 0, $\frac{a^k}{a^k+2} + \frac{a^{1-k}}{a^{1-k}+2}$ is always 1, so the sum of each pair's terms is 1, so $S = 999 + 0.5 = \boxed{999.5}$.